# Toric surfaces, vanishing Euler characteristic and Euler obstruction of a function

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In this work we define the vanishing Euler characteristic of a normal toric surface $X_\sigma$, we give a very simple formula to compute it, and we relate this number with the second polar multiplicity of $X_\sigma$. We also compute the Euler obstruction of a function $f : X_\sigma \to \mathbb{C}$ and the Brasselet number of it. As an application of this formula we compute the Euler obstruction of a type of polynomial on a family of determinantal surfaces.

## 1. INTRODUCTION

One of the most appealing aspects of toric varieties is the way that many questions that are difficult for general varieties, admit simple and concrete solutions in the toric case. The problem of finding resolutions of singularities is a perfect example ([12]).

In this paper we will work with the specific cases of toric surfaces and functions on toric surfaces. Let $X_\sigma$ be a toric surface with isolated singularity associated to the cone $\sigma \subset \mathbb{R}^2$, this type of singular surface has many special properties, one of them, that we do not find in general, is that this singularity admits a smoothing associated to its minimal resolution ([25]).

Let $X(\Delta)$ be the generic fiber of this smoothing, based on [23] we define the vanishing Euler characteristic of $X_\sigma$ denoted by $\nu(X_\sigma)$. Using continued fractions techniques, we give a very simple formula to compute this number and we also prove that $\nu(X_\sigma)$ is related to the polar multiplicities.

Since that $X_\sigma$ is a normal singularity, it follows from a result of Greuel and Steenbrink ([16]) that $\beta_1(X(\Delta)) = 0$, where $\beta_1$ is the first Betti number, and how $X(\Delta)$ always has the homotopy type of a finite $CW$-complex of dimension $\leq 2$ we have that $H_i(X(\Delta)) = 0$ if $i > 2$, then $\dim H_2(X(\Delta)) = \chi(X(\Delta)) - 1$. Therefore, the vanishing Euler characteristic of $X_\sigma$ is equal the middle Betti number of $X(\Delta)$, then in the case that $X_\sigma$ has a unique smoothing, this number coincides with an important invariant in singularity theory, called Milnor number.

The Milnor number was defined by Milnor in [21]. Initially this invariant was associated to germs of analytic functions $f : (X, 0) \to (\mathbb{C}, 0)$ with isolated singularity, and to study isolated hypersurfaces singularities. However this invariant is well defined in many others contexts, for instance curves ([8]), isolated complete intersection singularities, or ICIS ([17],

[19]) and determinantal varieties with codimension two ([24]). When $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a germ of an analytic function with isolated singularity at the origin, an important characteristic of the Milnor number of $f$, that we denote by $\mu(f)$, is that this invariant coincides with the number of Morse points of a Morsification of $f$.

Now, give $(X, 0)$ a germ of an analytic singular space embedded in $\mathbb{C}^n$ and $f : (X, 0) \to (\mathbb{C}, 0)$ a germ of analytic function with isolated singularity at the origin, in this situation, Brasselet, Massey, Parameswaran and Seade introduced an invariant associated to $f$ called the Euler obstruction of $f$ ([5]), in [30] the authors proved that the Euler obstruction of $f$ is, up to sign, the number of Morse points of a Morsification of $f$ on the regular part of $X$. Hence this invariant can be seen as a generalization of the Milnor number of $f$. In the last section, we give some interesting formula to compute the Euler obstruction of a function $f : X_\sigma \to \mathbb{C}$ with isolated singularity at 0, and also for another generalization of the Euler obstruction called Brasselet number ([9]).

## 2. BACKGROUND MATERIAL

In this section, we present different mathematical notions that we use in order to establish our results in the next sections. We recall first some important objects developed to study the structure of toric surfaces. We describe their singularities and also present two interesting examples.

### 2.1. Toric surfaces

For an overview about toric varieties see [12].

Definition 2.1. Let $\sigma \subset \mathbb{R}^2$ be a rational, strongly convex, polyhedral cone and let

$$\check{\sigma} = \left\{ v \in \mathbb{R}^2 ; \langle a, v \rangle \geq 0, \forall a \in \sigma \right\}$$

be the dual cone. The corresponding lattices $\mathbb{Z}^2$ are denoted by $N \subset \mathbb{R}^2$ and $M \subset \mathbb{R}^2$, respectively. Then the affine toric surface is defined as $X_\sigma = Spec\,\mathbb{C}[\check{\sigma} \cap M]$.

*Remark 2. 1.* As generally known, the affine toric surfaces are exactly those affine, normal surfaces admitting a $(\mathbb{C}^*)^2$-action with an open, dense orbit.

A strongly convex cone in $\mathbb{R}^2$ has the following normal form that will simplify our study of the singularities of toric surfaces.

Proposition 2.1. *Let $\sigma \subset \mathbb{R}^2$ be a strongly convex cone, then $\sigma \subset \mathbb{R}^2$ is the cone generated by vectors $v_1 = pe_1 - qe_2$ and $v_2 = e_2$, for some integers $p, q \in \mathbb{Z}_{>0}$ such that $0 < q < p$ and $p, q$ are coprime.*

Given a cone $\sigma \subset \mathbb{R}^2$, Riemenschneider proved in [25, 26] that the binomials which generate the ideal $I_\sigma$ are given by *quasiminors* of a *quasimatrix*, where $X_\sigma = V(I_\sigma)$. In the following we recall the definition of *quasimatrix*.

Definition 2.2. Given $A_i, B_i, C_{l,l+1} \in \mathbb{C}$ with $i = 1, \ldots, k$ and $l = 1, \ldots, k-1$, a quasimatrix with entries $A_i, B_i, C_{l,l+1}$ is written as

$$A = \begin{pmatrix} A_1 & & A_2 & \cdots & A_{k-1} & & A_k \\ B_1 & & B_2 & \cdots & B_{k-1} & & B_k \\ & C_{1,2} & & \cdots & & C_{k-1,k} & \end{pmatrix}.$$

The quasiminors of quasimatrix $A$ are defined by $A_iB_j - B_i(C_{i,i+1} \ldots C_{j-1,j})A_j$ for $1 \leq i < j \leq l$.

Given a cone $\sigma \subset \mathbb{R}^2$ generated by vectors $v_1 = pe_1 - qe_2$ and $v_2 = e_2$, where $0 < q < p$ and $p, q$ are coprime, consider the Hirzebruch-Jung continued fraction

$$\frac{p}{p-q} = a_2 - \frac{1}{a_3 - \frac{1}{\cdots - \frac{1}{a_{n-1}}}} = [[a_2, a_3, \ldots, a_{n-1}]]$$

where the integers $a_2, \ldots, a_{n-1}$ satisfies $a_i \geq 2$, for $i = 2, \ldots, n-1$. In [26] Riemenschneider proved the following:

Proposition 2.2. *The ideal $I_\sigma$ is generated by the quasiminors of the quasimatrix*

$$\begin{pmatrix} z_1 & & z_2 & & z_3 & \cdots & z_{n-2} & & z_{n-1} \\ z_2 & & z_3 & & z_4 & \cdots & z_{n-1} & & z_n \\ & z_2^{a_2-2} & & z_3^{a_3-2} & & \cdots & & z_{n-1}^{a_{n-1}-2} & \end{pmatrix}.$$

*Where the $a_i$ are given by the Hirzebruch-Jung continued fraction of $\frac{p}{p-q}$. Moreover, this set of generators is minimal.*

Then, if $a_i = 2$ for $i = 3, \ldots, n-2$, we have that $X_\sigma$ is a determinantal surface ([24]), in particular if the minimal dimension of embedding of $X_\sigma$ is 4, *i. e.*, if

$$\frac{p}{p-q} = a_2 - \frac{1}{a_3}$$

then $X_\sigma$ is always determinantal and the ideal $I_\sigma$ is generated by the $2 \times 2$ minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3^{a_3-1} \\ z_2^{a_2-1} & z_3 & z_4 \end{pmatrix}.$$

Example 2.1. Let $X_\sigma \subset \mathbb{C}^4$ be the toric surface associated to the cone $\sigma \subset \mathbb{R}^2$ generated by vectors $v_1 = e_2$ and $v_2 = 14e_1 - 11e_2$. From the Hirzebruch-Jung continued fraction process we have

$$\frac{14}{3} = 5 - \frac{1}{3},$$

then $X_\sigma = V(I_\sigma)$ where $I_\sigma$ is the ideal generated by the $2\times 2$ minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3^2 \\ z_2^4 & z_3 & z_4 \end{pmatrix}$$

*i.e.*, $X_\sigma$ is a codimension 2 determinantal surface.

Example 2.2. Let $X_\sigma \subset \mathbb{C}^5$ be the toric surface associated to the cone $\sigma \subset \mathbb{R}^2$ generated by vectors $v_1 = e_2$ and $v_2 = 4e_1 - e_2$. From the Hirzebruch-Jung continued fraction process we have

$$\frac{4}{3} = 2 - \frac{1}{2-\frac{1}{2}},$$

then $X_\sigma = V(I_\sigma)$ where $I_\sigma$ is the ideal generated by the $2\times 2$ minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & z_3 & z_4 & z_5 \end{pmatrix}$$

*i.e.*, $X_\sigma$ is a codimension 3 determinantal surface.

### 2.2. The Euler obstruction and Applications

An important object used in this work is the Euler obstruction, that was defined by MacPherson in [20] as a tool to prove the conjecture about existence and unicity of the Chern classes in the singular case. The Euler obstruction was deeply investigated by many authors, and for an overview about it see [2]. Let us now introduce some objects in order to define the Euler obstruction.

Let $(X,0) \subset (\mathbb{C}^n,0)$ be an equidimensional reduced complex analytic germ of dimension $d$ in an open set $U \subset \mathbb{C}^n$. We consider a complex analytic Whitney stratification $\mathcal{V} = \{V_i\}$ of $U$ adapted to $X$ and we assume that $\{0\}$ is a stratum. We choose a small representative of $(X,0)$ such that 0 belongs to the closure of all the strata. We will denote it by $X$ and we will write $X = \cup_{i=0}^q V_i$ where $V_0 = \{0\}$ and $V_q = X_{\text{reg}}$ is the set of regular points of $X$. We will assume that the strata $V_0, \ldots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \ldots, \overline{V_{q-1}}$ are reduced.

Let $G(d,n)$ denote the Grassmanian of complex $d$-planes in $\mathbb{C}^n$. On the regular part $X_{\text{reg}}$ of $X$ the Gauss map $\phi : X_{\text{reg}} \to U \times G(d,n)$ is well defined by $\phi(x) = (x, T_x(X_{\text{reg}}))$.

Definition 2.3. The *Nash transformation* (or *Nash blow up*) $\widetilde{X}$ of $X$ is the closure of image $\text{Im}(\phi)$ in $U \times G(d,n)$. It is a (usually singular) complex analytic space endowed with an analytic projection map $\nu : \widetilde{X} \to X$ which is a biholomorphism away from $\nu^{-1}(\text{Sing}(X))$.

The fiber of the tautological bundle $\mathcal{T}$ over $G(d,n)$, at point $P \in G(d,n)$, is the set of vectors $v$ in the $d$-plane $P$. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d,n)$. Let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{X}$, with projection map $\pi$. The bundle $\widetilde{T}$ on $\widetilde{X}$ is called *the Nash bundle* of $X$.

An element of $\widetilde{T}$ is written $(x, P, v)$ where $x \in U$, $P$ is a $d$-plane in $\mathbb{C}^n$ based at $x$ and $v$ is a vector in $P$. We have the following diagram:

$$\begin{array}{ccc} \widetilde{T} & \hookrightarrow & \mathcal{T} \\ \pi\downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & U \times G(d,n) \\ \nu\downarrow & & \downarrow \\ X & \hookrightarrow & U. \end{array}$$

Adding the stratum $U \setminus X$ we obtain a Whitney stratification of $U$. Let us denote by $TU|_X$ the restriction to $X$ of the tangent bundle of $U$. We know that a stratified vector field $v$ on $X$ means a continuous section of $TU|_X$ such that if $x \in V_\alpha \cap X$ then $v(x) \in T_x(V_\alpha)$. By Whitney condition one has the following lemma ([6]).

Lemma 2.1. *Every stratified vector field $v$ on a subset $A \subset X$ has a canonical lifting to a section $\tilde{v}$ of the Nash bundle $\tilde{T}$ over $\nu^{-1}(A) \subset \tilde{X}$.*

Now consider a stratified radial vector field $v(x)$ in a neighborhood of $\{0\}$ in $X$, *i.e.*, there is $\varepsilon_0$ such that for every $0 < \varepsilon \leq \varepsilon_0$, $v(x)$ is pointing outwards the ball $B_\varepsilon$ over the boundary $S_\varepsilon := \partial B_\varepsilon$.

The following interpretation of the local Euler obstruction has been given by Brasselet and Schwartz in [6].

Definition 2.4. Let $v$ be a radial vector field on $X \cap S_\varepsilon$ and $\tilde{v}$ the lifting of $v$ on $\nu^{-1}(X \cap S_\varepsilon)$ to a section of the Nash bundle. The local Euler obstruction (or simply the Euler obstruction) $\mathrm{Eu}_X(0)$ is defined to be the obstruction to extending $\tilde{v}$ as a nowhere zero section of $\tilde{T}$ over $\nu^{-1}(X \cap B_\varepsilon)$.

More precisely, let $\mathcal{O}(\tilde{v}) \in H^{2d}(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$ be the obstruction cocycle to extending $\tilde{v}$ as a nowhere zero section of $\tilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$. The local Euler obstruction $\mathrm{Eu}_X(0)$ is defined as the evaluation of the cocycle $\mathcal{O}(\tilde{v})$ on the fundamental class of the pair $((X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$. The Euler obstruction is an integer.

In [4], Brasselet, Lê and Seade give a Lefschetz type formula for the local Euler obstruction. The formula shows that the local Euler obstruction, as a constructible function, satisfies the Euler condition relatively to generic linear forms.

Theorem 2.1. *Let $(X, 0)$ and $\{V_i\}$ be given as before, then for each generic linear form $l$, there is $\varepsilon_0$ such that for any $\varepsilon$ with $0 < \varepsilon < \varepsilon_0$ and $t_0 \neq 0$ sufficiently small, the Euler obstruction of $(X, 0)$ is equal to:*

$$\mathrm{Eu}_X(0) = \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap l^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i),$$

*where $\chi$ denotes the Euler-Poincaré characteristic,* $\mathrm{Eu}_X(V_i)$ *is the value of the Euler obstruction of $X$ at any point of $V_i$, $i = 1, \dots, q$, and $0 < |\delta| \ll \varepsilon \ll 1$.*

In the last section we will study two generalizations of the Euler obstruction, the Euler obstruction of a function, defined in [5], and the Brasselet number, defined in [9]. Then let us recall these two definitions.

Introduced by Brasselet, Massey, Parameswaran and Seade in [5], the Euler obstruction of a function measures in a way how far the equality given in Theorem 2.1 is from being true if we replace the generic linear form $l$ with some other function on $X$ with at most an isolated stratified critical point at 0. Let $f : X \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $F : U \to \mathbb{C}$. A point $x$ in $X$ is a critical point of $f$ if it is a critical point of $F_{|V(x)}$, where $V(x)$ is the stratum containing $x$. We assume that $f$ has an isolated singularity (or an isolated critical point) at 0, *i.e.*, that $f$ has no critical point in a punctured neighborhood of 0 in $X$. In order to define the new invariant the authors constructed a stratified vector field on $X$, denoted by $\overline{\nabla}_X f$. This vector field is homotopic to $\overline{\nabla} F|_X$ and one has $\overline{\nabla}_X f(x) \neq 0$ unless $x = 0$.

Let $\tilde{\zeta}$ be the lifting of $\overline{\nabla}_X f$ as a section of the Nash bundle $\widetilde{T}$ over $\widetilde{X}$ without singularity over $\nu^{-1}(X \cap S_\varepsilon)$. Let $\mathcal{O}(\tilde{\zeta}) \in H^{2d}\big(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon)\big)$ be the obstruction cocycle to the extension of $\widetilde{\zeta}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$.

Definition 2.5. The local Euler obstruction $\mathrm{Eu}_{f,X}(0)$ is the evaluation of $\mathcal{O}(\tilde{\zeta})$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$.

The following result compares the Euler obstruction of the space $X$ with the Euler obstruction of a function on $X$ ([5]).

Theorem 2.2. *Let $(X, 0)$ and $\{V_i\}$ given as before and let $f : (X, 0) \to (\mathbb{C}, 0)$ be a function with an isolated singularity at 0. For $0 < |\delta| \ll \varepsilon \ll 1$ we have:*

$$\mathrm{Eu}_{f,X}(0) = \mathrm{Eu}_X(0) - \left( \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap f^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i) \right).$$

For an overview about Euler obstruction of a function see [3].

In [9] Dutertre and Grulha defined the Brasselet number $B_{f,X}(0)$, even in the general case this definition involves some technical elements, when $f$ has an isolated singularity, this number is equal to the difference $\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0)$.

### 2.3. The Generic Fiber

In this section we remember the definition of smoothing of an analytic variety and some important results related to it.

Let $X_0 \subset \mathbb{C}^n$ be a germ of an analytic $d$-dimensional variety, on some open set of $\mathbb{C}^n$ with isolated singularity at the origin. A smoothing of $X_0$ is a flat deformation with the property that its generic fiber is smooth. More precisely:

Definition 2.6. We say that a germ of analytic variety $(X_0, 0)$ with isolated singularity of complex dimension $d \geq 1$ has a smoothing, if there exist an open ball $B_\varepsilon(0) \subset \mathbb{C}^n$ centered at the origin, a closed subspace $X \subset B_\varepsilon(0) \times D$, where $D \subset \mathbb{C}$ is an open disc with center at zero and a proper analytic map $F : X \longrightarrow D$, with the restriction to $X$ of the projection $p : B_\varepsilon(0) \times D \longrightarrow D$ such that

a) $F$ is flat;
b) $(F^{-1}(0), 0)$ is isomorphic to $(X_0, 0)$;
c) $F^{-1}(t)$ is non singular for $t \neq 0$.

It follows from the above definition that $X$ has isolated singularity at the origin and it is a normal variety if $X_0$ is normal at zero. Moreover,

$$F\,|_{F^{-1}(D-\{0\})}\colon F^{-1}(D - \{0\}) \longrightarrow D - \{0\}$$

is a fiber bundle whose fibers $X_t = F^{-1}(t)$ are non singular.

The topology of the generic fiber has been intensively studied. For determinantal variety, for instance, the following result was proved by Wahl in [32].

Theorem 2.3. *Let $(X, 0)$ be a determinantal variety with isolated singularity at the origin defined by $t \times t$ minors of an $s \times p$ matrix $M$, whose entries are on the ring of convergent power series on $\mathbb{C}^n$, and $2 \leq t \leq s \leq p$. If $dim(X) < s + p - 2t + 3$, then $X$ has a smoothing.*

In particular, it follows from this result that if $(X, 0)$ is Cohen-Macaulay with codimension less than or equal to 2 and $dim(X, 0) \leq 3$, then $(X, 0)$ admits a smoothing. We also observe that for Cohen-Macaulay singularities of codimension less than or equal to 2, there is no obstruction for lifting second-order deformations, the basis of the semi-universal deformation is smooth ([11]).

The following result was proved by Greuel and Steenbrink in [16].

Theorem 2.4. *Let $X_t$ be the Milnor fiber of a smoothing of a normal singularity, then* $\beta_1(X_t) = 0$.

In [10] Ebeling and Gusein-Zade introduced the definition of a determinantal variety with an *essentially isolated determinantal singularity* (EIDS), and they observed that, if an EIDS $X \subset \mathbb{C}^n$ is defined by $t \times t$ minors of an $s \times p$ matrix $M$, then $X$ has an isolated singularity at the origin if, and only if, $n \leq (s - t + 2)(p - t + 2)$. Moreover, according to the Thom Transversality Theorem an EIDS always has an *essential smoothing*, and in the specific case that $n < (s - t + 2)(p - t + 2)$ the *essential smoothing* is a genuine smoothing. The topology of the general fiber of a smoothing of an isolated determinantal variety was also studied by Ballesteros, Okamoto and Tomazella in [23].

*Remark 2. 2.* A very important fact is that not all variety $X$ has a smoothing, for instance, if $X = X_\sigma$ is a $d$-dimensional toric variety, where $\sigma \subset \mathbb{R}^d$ is a simplicial cone, then $X_\sigma$ is a quotient singularity. In [28] Schlessinger proved that all quotient singularity $X$ with $\dim X \geq 3$ is rigid, it means that $X$ has no nontrivial first order deformations. Therefore, if $d \geq 3$ then $X_\sigma$ has no nontrivial first order deformations.

## 3. THE VANISHING EULER CHARACTERISTIC OF A TORIC SURFACE

Throughout this section we denote by $X_\sigma \subset \mathbb{C}^n$ a toric surface. Using resolution of singularities and Hirzebruch-Jung continued fraction we obtain a refinament $\Delta$ of $\sigma$ and we construct a special smoothing of $X_\sigma$. Thus we can define a vanishing Euler characteristic of $X_\sigma$.

In order to resolve the singularity of $X_\sigma$, we put $\sigma$ into standard form (generated by $e_2$, $pe_1 - qe_2$, $0 \leq q < p$ with $gcd(p, q) = 1$). If $p = 1$ (so $q = 0$) then the surface is nonsingular (and corresponds to $\mathbb{C}^2$); otherwise, we insert ray $e_1$ (this is a blow up at the fixed point) since the cone generated by $e_1$, $e_2$ will be nonsingular and the lower cone will have a singular point which is "less" singular than the original one. To see this, we can position this smaller cone to standard form by a rotation of an angle $\frac{\pi}{2}$ of the lattice (moving $e_1$ to $e_2$) and then translating the other vector vertically to put it in position $(p_1, -q_1)$ with $p_1 = q$, $0 \leq q_1 < p_1$ and $q_1 = b_1 q - p$ for some integer $b_1 \geq 2$.

This corresponds to a smooth cone when $q_1 = 0$; otherwise

$$\frac{p}{q} = b_1 - \frac{q_1}{p_1} = b_1 - \frac{1}{\frac{p_1}{q_1}}$$

and the process can be repeated. We recognize this immediately as the Hirzebruch-Jung continued fraction for $p/q = [[b_1, \ldots, b_r]]$. We can see that there are $r$ added vertices $v_1, \ldots, v_r$ between the given vertices $v_0 = e_2$, $v_{r+1} = pe_1 - qe_2$ and $v_{i+1} = b_i v_i - v_{i-1}$.

In [12] Fulton provides the following proposition.

Proposition 3.1. *If $\Delta$ is the subdivision of $\sigma$ obtained by $v_i$'s then $X(\Delta)$ is the minimal equivariant resolution of singularities of $X_\sigma$.*

*Remark 3. 1.* A toric surface $X_\sigma$, which is a cyclic quotient singularity, always possesses one smoothing which is locally diffeomorphic to its resolution $X(\Delta)$ (see [25], Satz 10). The problem is that this smoothing need not be unique. For instance, in [32] Wahl gave an example of a smoothing for the toric surface $X_\sigma \subset \mathbb{C}^5$ associated to the cone $\sigma \subset \mathbb{R}^2$ generated by vectors $v_1 = e_2$ and $v_2 = 4e_1 - e_2$, whose fiber $X_t$ has Euler characteristic equal to 1. But in [23] the authors gave an example of a smoothing of $X_\sigma$ such that $\chi(X_t) = 2 = \chi(X(\Delta))$.

If $X_\sigma \subset \mathbb{C}^n$ is an ICIS (when $p = q + 1$), $X(\Delta)$ can be seen as the Milnor fiber of $X_\sigma$. In this case, $X(\Delta)$ has the homotopy type of a bouquet of spheres and the Milnor number of $X_\sigma$ is the number of such spheres (see [17]). In this case, the Milnor number coincides with the so-called vanishing Euler characteristic, that is

$$\mu(X_\sigma) = \beta_2(X(\Delta)) = \chi(X(\Delta)) - 1$$

where $\beta_2(X(\Delta))$ denotes the second Betti number of $X(\Delta)$. Based on [23] and supported by the previous remark, we make the following definition now in the general case of a toric surface.

Definition 3.1. We define the vanishing Euler characteristic of a toric surface $X_\sigma$ by

$$\nu(X_\sigma) := \chi(X(\Delta)) - 1.$$

Theorem 3.1. *Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and $p, q$ are coprimes, then*

$$\nu(X_\sigma) = (a_2 - 2) + \cdots + (a_{n-2} - 2) + (a_{n-1} - 1),$$

*where $a_2, \ldots, a_{n-1}$ are the integers coming from the Hirzebruch-Jung continued fraction of $\frac{p}{p-q}$.*

*Proof.* We know that

$$\chi(X(\Delta)) = \beta_0(X(\Delta)) - \beta_1(X(\Delta)) + \beta_2(X(\Delta)) = 1 - \beta_1(X(\Delta)) + \beta_2(X(\Delta)).$$

Since $X_\sigma$ is normal, by Theorem 2.4, $\beta_1(X(\Delta)) = 0$, then

$$\beta_2(X(\Delta)) = \chi(X(\Delta)) - 1 = \nu(X_\sigma).$$

By [1], we have

$$\dim H_2^{cld}(X(\Delta)) = d_1 - 2,$$

where $d_1$ denotes the number of 1-dimensional cones in $\Delta$. But, $X(\Delta)$ is a smooth surface, then

$$H_2^{cld}(X(\Delta)) = H_2(X(\Delta)),$$

*i.e.*, $\beta_2(X(\Delta)) = d_1 - 2 = r$. As a consequence of [25, 26], we have

$$\sum_{i=1}^{r} (b_i - 1) = \sum_{j=2}^{n-1} (a_j - 1),$$

so

$$r = (a_2 - 2) + \cdots + (a_{n-2} - 2) + (a_{n-1} - 1).$$

∎

In particular, if a variety $X$ of dimension $d$ has a unique smoothing the Milnor number of $X$ is defined as the $d$th Betti number $\beta_d(X_t)$ of the generic fiber $X_t$ of a smoothing of $X$, whenever $X_t$ has homology only in the middle dimension, that is

$$\mu(X) := \beta_d(X_t).$$

Thus, we have the following consequence.

Corollary 3.1. *If $X_\sigma$ is a toric surface that admits a unique smoothing, then*

$$\nu(X_\sigma) = \beta_2(X(\Delta)) = \mu(X_\sigma).$$

Example 3.1. Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and $p, q$ are coprimes, such that

$$\frac{p}{p-q} = a - \frac{1}{b},$$

then by Theorem 3.1

$$\nu(X_\sigma) = (a-2) + (b-1).$$

But, from [25, 26] we know that $X_\sigma$ is a determinantal surface in $\mathbb{C}^4$ given by the $2 \times 2$ minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3^{b-1} \\ z_2^{a-1} & z_3 & z_4 \end{pmatrix}$$

and in [24] Pereira and Ruas proved that when $X \subset \mathbb{C}^4$ is a determinantal surface $X$ has a unique smoothing. The authors also present a formula to compute the Milnor number in this case, that coincides with our formula to compute $\nu$.

Given $X$ an isolated singularity smoothable we know that $\chi(X_t)$ is independent of $t$. When we consider $v$ a radial continuous vector field on $X$ with isolated singularity at 0, we can relate the number $\chi(X_t)$ with the GSV index of $v$ in $X$. The GSV index was introduced by Gómez-Mont, Seade and Verjovsky in [14, 28] for hypersurface germs, and extended in [29] to complete intersections. In Section 3 of [7] we can find the definition of this index for the case that $X$ admits a smoothing, which depends of the smoothing given by $F$. They also proved that $\mathrm{Ind}_{GSV}(v, X, F) = \chi(X_t)$, then we have the following.

Corollary 3.2. *Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and $p, q$ are coprimes, and consider $v$ a radial continuous vector field on $X_\sigma$ with isolated singularity at* 0. *Then,*

$$\mathrm{Ind}_{GSV}(v, X_\sigma, \mathrm{Res}) = (a_2 - 2) + \cdots + (a_{n-2} - 2) + (a_{n-1}),$$

*where* $a_2, \ldots, a_{e-1}$ *are the integers coming from the Hirzebruch-Jung continued fraction of* $\frac{p}{p-q}$ *and* $\mathrm{Ind}_{GSV}(v, X_\sigma, \mathrm{Res})$ *is the GSV index of* $v$ *in* $X_\sigma$ *relative to the smoothing whose fiber is* $X(\Delta)$*.*

*Remark 3. 2.* In [23] the authors defined the Milnor number of a function $f$ on an Isolated Determinantal Singularity, or IDS $X$, by the number of critical points of a specific function $f_a$ defined in a fiber $X_A$ of a smoothing of $X$. But, by definition this number is equal to

$$\mathrm{Ind}_{GSV}(v, X, F)$$

where $v$ is the vector field given by the gradient of the function $f$ and $F$ is the flat map associated to the smoothing of $X$ whose fiber is $X_A$. Then, the Milnor number of Ballesteros, Okamoto and Tomazella can be extended to the case of toric surfaces.

## 4. RELATION WITH POLAR MULTIPLICITIES AND EULER OBSTRUCTION

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a $d$-dimensional variety with isolated singularity at the origin. Suppose that $X$ has a smoothing, that is, there exists a family $\Pi : \mathfrak{X} \to D \subset \mathbb{C}$, restriction of the projection $\Phi : B_\varepsilon(0) \times D \to D$, such that $X_t = \Pi^{-1}(t)$ is smooth for all $t \neq 0$ and $X_0 = X$.

The variety $\mathfrak{X}$ also has isolated singularity at the origin. Let $p$ be a complex analytic function defined in $X$ with isolated singularity at the origin. Let

$$\begin{array}{rcl} \tilde{p} : \mathfrak{X} \subset \mathbb{C}^n \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ (x, t) & \longmapsto & \tilde{p}(x, t) \end{array}$$

such that $\tilde{p}(x, 0) = p(x)$ and for all $t \neq 0$, $\tilde{p}(., t) = p_t$ is a Morse function in $X_t$. Thus we have the following diagram

$$\begin{array}{ccc} X_t \subset & \mathfrak{X} & \subset \mathbb{C}^n \times \mathbb{C} \\ \downarrow_{p_t} & & \downarrow_{(\Pi, p)} \\ \mathbb{C} \times \{t\} & & \mathbb{C} \times \mathbb{C} \end{array} \tag{4.1}$$

Notice that the number of critical points of $p_t$ is finite. In fact, $x$ is a critical point of $p_t$ if and only if $x$ is a critical point of the function $Re(p_t) : X_t \to \mathbb{R}$. Since the real part of $p_t$ is an analytic function on $X_t$, the number of critical points of $Re(p_t)$ and, hence of $p_t$, is finite. In [24] the authors proved the following.

Proposition 4.1. *Let* $X$ *be a* $d$*-dimensional variety with isolated singularity at the origin admitting a smoothing and* $p_t : X_t \to \mathbb{C}$, $p_t = \tilde{p}(., t)$ *as above. Then,*

$$\chi(X_t) = \chi(p_t^{-1}(0)) + (-1)^d n_\sigma$$

*where $n_\sigma$ is the number of critical points of $p_t$ and $\chi(X_t)$ denotes the Euler characteristic of $X_t$.*

The above formula can also be expressed replacing $n_\sigma$ by $m_d(X)$ the $d$-th polar multiplicity of $X$. We refer to [31] for the definition and properties of polar varieties. Here, we follow [13] to define the $d$-th polar multiplicity.

Let $\mathfrak{X} \subset \mathbb{C}^n \times \mathbb{C}^s$ be a complex analytic variety of complex dimension $d+s$ and $\Pi : \mathfrak{X} \to \mathbb{C}^s$ an analytic function such that $\Pi^{-1}(0) = X$. Let $\tilde{p} : \mathfrak{X} \subset \mathbb{C}^n \times \mathbb{C}^s, 0 \to \mathbb{C}^s, 0$ be such that $\tilde{p}|_X$ has isolated singularity at the origin. Then, we can define $m_d(X, \tilde{p}, \Pi) = m_0(P_d(\Pi, \tilde{p}))$, where $P_d(\Pi, \tilde{p})$ is the polar variety of $\mathfrak{X}$ with respect to $(\Pi, \tilde{p})$.

In general, $m_d(X, \tilde{p}, \Pi)$ depends on the choices of $\mathfrak{X}$ and $\tilde{p}$, but when $\mathfrak{X}$ is a versal deformation of $X$ or in the case that $X$ has a unique smoothing, $m_d$ depends only on $X$ and $\tilde{p}$. Furthermore, if $\tilde{p}$ is a generic linear projection, $m_d$ is an invariant of the analytic variety $X$, which we denote by $m_d(X)$.

When $s = 1$ and $\tilde{p}$ is a generic linear projection, we recover the conditions in diagram (4.1) and we can relate $n_\sigma$ and $m_d(X)$. In fact, the following result is a direct consequence of the definitions of these two invariants.

Proposition 4.2. *Under the conditions of Proposition 4.1, $n_\sigma = m_d(X)$.*

Theorem 4.1. *Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and $p, q$ are coprimes, then*

$$m_2(X_\sigma) = (a_2 - 1) + \cdots + (a_{n-2} - 1) + (a_{n-1}),$$

*where $a_2, \ldots, a_{n-1}$ are the integers coming from the Hirzebruch-Jung continued fraction of $\frac{p}{p-q}$.*

*Proof.* Let $p : X_\sigma \to \mathbb{C}$ be a generic linear function, then by Propositions 4.1 and 4.2

$$\chi(X(\Delta)) = \chi(p_t^{-1}(0)) + (-1)^d n_\sigma = \chi(X(\Delta) \cap p_t^{-1}(0)) + m_2(X_\sigma),$$

but $X(\Delta) \cap p_t^{-1}(0)$ is homeomorphic to $X \cap p^{-1}(c)$, since

$$f := (\Pi, \tilde{p}) : \mathfrak{X} \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$

is a bundle in the punctured disk and

$$X(\Delta) \cap p_t^{-1}(0) = f^{-1}(t, 0) \quad \text{and} \quad X \cap p^{-1}(c) = f^{-1}(0, c)$$

are fibers of this fibration. Then

$$\chi(X(\Delta)) = \chi(X \cap p^{-1}(c)) + m_2(X_\sigma)$$

but by [4] we have $\chi(X \cap p^{-1}(c)) = \mathrm{Eu}_{X_\sigma}(0)$. In [15] Gonzalez-Sprinberg proved that $\mathrm{Eu}_{X_\sigma}(0) = 3 - n$, then by Theorem 3.1 we have

$$m_2(X_\sigma) = (a_2 - 1) + \cdots + (a_{n-2} - 1) + (a_{n-1}).$$

∎

Next, we will give an illustration, with an example, of how easy it is to compute $m_2$ using Theorem 4.1. The next example was also computed in [10, 23, 24], but here this computation is done in an easier way. For instance, in [23] they used a software to compute this invariant and here we use only the continued fractions.

Example 4.1. Let $X_\sigma \subset \mathbb{C}^4$ be the toric surface associated to the cone $\sigma \subset \mathbb{R}^2$ generated by vectors $v_1 = e_2$ and $v_2 = 3e_1 - e_2$. From the Hirzebruch-Jung continued fraction process we have

$$\frac{3}{2} = 2 - \frac{1}{2},$$

then $X_\sigma = V(I_\sigma)$ where $I_\sigma$ is the ideal generated by the $2 \times 2$ minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}$$

*i.e.*, $X_\sigma$ is a codimension 2 determinantal surface. Then,

$$m_2(X_\sigma) = (a_2 - 1) + a_3 = 3.$$

In a more general way, we have the following.

Corollary 4.1. *Consider $Y \subset \mathbb{C}^{n+1}$ the determinantal surface given by the $2 \times 2$ minors of the matrix*

$$A = \begin{pmatrix} z_1 & z_2 & \dots & z_{n-1} & z_n^b \\ z_2^a & z_3 & \dots & z_n & z_{n+1} \end{pmatrix},$$

*where $n \geq 2$ and $a, b$ are positive integers. Then, $m_2(Y) = a + b + n - 2$.*

*Proof.* In [25, 26] Riemenschneider proved that $Y = X_\sigma$, where $\sigma \subset \mathbb{R}^2$ is the cone generated by vectors $v_1 = e_2$ and $v_2 = w_{n+1}e_1 - u_{n+1}e_2$ with

$$\begin{aligned}
u_1e_1 + w_1e_2 &= e_1, \\
u_2e_1 + w_2e_2 &= e_1 + e_2, \\
u_3e_1 + w_3e_2 &= ((a+1)u_2 - u_1)e_1 + ((a+1)w_2 - w_1)e_2, \\
u_4e_1 + w_4e_2 &= (2u_3 - u_2)e_1 + (2w_3 - w_2)e_2, \\
u_5e_1 + w_5e_2 &= (2u_4 - u_3)e_1 + (2w_4 - w_3)e_2, \\
&\vdots \\
u_ne_1 + w_ne_2 &= (2u_{n-1} - u_{n-2})e_1 + (2w_{n-1} - w_{n-2})e_2, \\
u_{n+1}e_1 + w_{n+1}e_2 &= ((b+1)u_n - u_{n-1})e_1 + ((b+1)w_n - w_{n-1})e_2.
\end{aligned}$$

*i.e.*, the integers coming from the Hirzebruch-Jung continued fraction of $\frac{w_{n+1}}{w_{n+1}-u_{n+1}}$ are

$$a_2 = a+1, \quad a_3 = 2, \ldots, \quad a_{n-1} = 2, \quad a_n = b+1.$$

Therefore, from Theorem 4.1 we have that $m_2(Y) = a + b + n - 2$. ▮

## 5. THE EULER OBSTRUCTION OF A FUNCTION ON A TORIC SURFACE

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an algebraic variety and let $f : (X, 0) \to (\mathbb{C}, 0)$ be a function with isolated singularity at the origin, as we said before the Euler obstruction of $f$ can be viewed as a generalization of the Milnor number of the function $f$, that usually is denoted by $\mu(f)$. In this section, we compute the Euler obstruction of $f$ on a toric surface $X_\sigma$.

From now on we will consider the following setup.

Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and $p, q$ are coprimes, and consider $a_2, \ldots, a_{n-1}$ the integers coming from the Hirzebruch-Jung continued fraction of $\frac{p}{p-q}$. We will denote by $\mathcal{O}_{(0,\ldots,0)}$, $\mathcal{O}_{(1,0,\ldots,0)}$, $\mathcal{O}_{(0,\ldots,0,1)}$ and $\mathcal{O}_{(1,\ldots,1)}$ the four orbits of the action $\tilde{\varphi} : (\mathbb{C}^*)^2 \times X_\sigma \longrightarrow X_\sigma$ given by

$$\tilde{\varphi}((t_1, t_2), (x_1, \ldots, x_n)) = (t_1x_1, t_1t_2x_2, t_1^{u_3}t_2^{v_3}x_3, \ldots, t_1^{u_n}t_2^{v_n}x_n)$$

where $\{(1, 0), (1, 1), (u_3, v_3), \ldots, (u_n, v_n)\}$ is the minimal basis of the monoid $\check{\sigma} \cap \mathbb{Z}^2$. Using Theorem 2.2 we can prove the following.

THEOREM 5.1. *Let $f : (X_\sigma, 0) \to (\mathbb{C}, 0)$ be a function with isolated singularity at the origin, then*

$$\mathrm{Eu}_{f,X_\sigma}(0) = 3 - n - \chi(\gamma) - \#B,$$

*where $\gamma$ is the curve in $\mathbb{C}^2$ given by $\gamma(t_1, t_2) = f(t_1, t_1t_2, t_1^{u_3}t_2^{v_3}, \ldots, t_1^{u_n}t_2^{v_n}) - t_0$ and $B = X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0) \cap \mathcal{O}_{(0,\ldots,0,1)}$, with $t_0 \neq 0$.*

*Proof.* Since $X_\sigma$ has isolated singularity at the origin, by Theorem 2.2

$$\mathrm{Eu}_{f,X_\sigma}(0) = \mathrm{Eu}_{X_\sigma}(0) - \chi\big(X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0)\big).$$

We know that the orbit $\mathcal{O}_{(1,\ldots,1)}$ of the action $\tilde{\varphi}$ is homeomorphic to $(\mathbb{C}^*)^2$. Now, consider the application

$$\varphi : \mathbb{C}^2 \to X_\sigma$$

given by $\varphi(t_1,t_2) = (t_1, t_1t_2, t_1^{u_3}t_2^{v_3}, \ldots, t_1^{u_n}t_2^{v_n})$, then $\varphi(\mathbb{C}^2) = X_\sigma \backslash \mathcal{O}_{(0,\ldots,0,1)}$. Furthermore,

$$\varphi|_{\mathbb{C}^*\times\mathbb{C}} : \mathbb{C}^* \times \mathbb{C} \to X_\sigma \setminus \big\{\mathcal{O}_{(0,\ldots,0,1)} \cup \mathcal{O}_{(0,\ldots,0)}\big\}$$

is a bijection. Then, consider the curve $\gamma : \mathbb{C}^2 \to \mathbb{C}$ given by

$$\gamma(t_1,t_2) = f(t_1, t_1t_2, t_1^{u_3}t_2^{v_3}, \ldots, t_1^{u_n}t_2^{v_n}) - t_0$$

we have that

$$\chi\big(X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0)\big) = \chi(\gamma) + \#A - \#(\varphi^{-1}(A) \cap \gamma) + \#B,$$

where $A = X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0) \cap \mathcal{O}_{(1,0,\ldots,0)}$ and $B = X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0) \cap \mathcal{O}_{(0,\ldots,0,1)}$. But it is easy to see that $\#A = \#(\varphi^{-1}(A) \cap \gamma)$, then

$$\chi\big(X_\sigma^{reg} \cap B_\varepsilon \cap f^{-1}(t_0)\big) = \chi(\gamma) + \#B.$$

By [15] we know that $\mathrm{Eu}_{X_\sigma}(0) = 3 - n$, therefore $\mathrm{Eu}_{f,X_\sigma}(0) = 3 - n - \chi(\gamma) - \#B$. ∎

EXAMPLE 5.1. Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $v_1 = e_2$ and $v_2 = ne_1 - e_2$, where $n > 1$, *i.e.*, $X_\sigma$ is the determinantal surface given by the $2 \times 2$ minors of the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\ z_2 & z_3 & z_4 & \cdots & z_n & z_{n+1} \end{pmatrix}$$

and consider $f : (X_\sigma, 0) \to (\mathbb{C}, 0)$ the polynomial with isolated singularity at the origin given by

$$f(x_1, \ldots, x_{n+1}) = x_1^{n+1} + x_{n+1} + \sum_{l=1}^{k} a_l x_2^{d_2^l} \ldots x_n^{d_n^l}$$

with $d_2^i, \ldots, d_n^i$ satisfying condition $n+1 = \sum_{j=2}^{n} d_j^1(u_j + v_j) = \cdots = \sum_{j=2}^{n} d_j^k(u_j + v_j)$ and $a_l \in \mathbb{C}^*$ for $l = 1, \ldots, k$. In this case, we know that

$$(u_3, v_3) = (1, 2), \quad (u_4, v_4) = (1, 3), \ldots, (u_{n+1}, v_{n+1}) = (1, n)$$

then the curve in $\mathbb{C}^2$

$$\tilde{\gamma}(t_1,t_2)=f(t_1,t_1t_2,t_1^{u_3}t_2^{v_3},\ldots,t_1^{u_n}t_2^{v_n})$$

is given by

$$\tilde{\gamma}(t_1,t_2)=t_1^{n+1}+t_1t_2^n+\sum_{l=1}^{k}a_l(t_1t_2)^{d_2^l}\ldots(t_1t_2^{n-1})^{d_n^l}$$

that is a homogeneous polynomial of degree $n+1$ with isolated singularity at the origin. Now, recall that curve $\gamma$ is the fiber of a smoothing of curve $\tilde{\gamma}$, then by [22] $\mu(\tilde{\gamma})=n^2$. Since $\chi(\gamma)=1-\mu(\tilde{\gamma})$, we have that $\chi(\gamma)=1-n^2$. Note yet that, $\#B=1$, then by Theorem 5.1

$$\mathrm{Eu}_{f,X_\sigma}(0)=n^2-n.$$

Our last result concern about the Brasselet number ([9]). This invariant is a generalization of the Euler obstruction and of the Euler obstruction of a function.

Corollary 5.1. *Let $f:X_\sigma\to\mathbb{C}$ be a function with isolated singularity at the origin, then*

$$B_{f,X_\sigma}(0)=\chi(\gamma)+\#B$$

*where $\gamma$ and $B$ are as above.*

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